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On Product of Generalized s-Convex Functions and New Inequalities on Fractal Sets

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ABSTRACT

In this article, we give some new inequalities for product of generalized s-convex functions on the co-ordinates on fractal sets. Furthermore, this article points out some of these kinds of inequalities.

Keywords: Co-ordinates, fractal space, local fractional derivative.

1. Introduction

Assume that $f\colon V \subset \mathbb{R} \longrightarrow \mathbb{R}^{\alpha}, \forall x_1, x_2 \in V$ and $\gamma \in [0,1]$ if the next inequality $f(\gamma x_1 + (1-\gamma)x_2) \leq \gamma^{\alpha} f(x_1) + (1-\gamma)^{\alpha} f(x_2)$ holds, then f is called a generalized convex function (GC) on V (Mo et al. (2014)) and when $\alpha = 1$, we have convex function (Hörmander (1994)).

Recently, researchers have payed attention to convexity of functions to solve some problems in economy, biological system and optimization, for instance, (see Grinalatt and Linnainmaa (2011), Ruel and Ayres (1999)).

Let $h \in {}_{\kappa_1}I_{\kappa_2}^{(\alpha)}$ be a GC on $[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$. Then

$$h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(\kappa_2 - \kappa_1)^{\alpha}}\ {}_{\kappa_1}I_{\kappa_2}^{(\alpha)} h(x) \leq \frac{h(\kappa_1) + h(\kappa_2)}{2^{\alpha}} \tag{1}$$

is known as generalized Hermite-Hadmard's inequality (Mo et al. (2014)). If $\alpha = 1$ in (1), then we have the classical Hermite-Hadmard's inequality (Dragomir and Fitzpatrick (1999)).

From (Mo and Sui (2014)), we state the next definition:

Definition 1.1. *A function $h\colon \mathbb{R}_+ \longrightarrow \mathbb{R}^{\alpha}$, is called generalized s-convex ($0 < s < 1$) in the first sense which is denoted by GK_s^1 (in the second sense which is denoted by GK_s^2) if*

$$h(\gamma_1 x_1 + \gamma_2 x_2) \leq \gamma_1^{s\alpha} h(x_1) + \gamma_2^{s\alpha} h(x_2), \tag{2}$$

$\forall x_1, x_2 \in \mathbb{R}_+$ *;* $\forall \gamma_1, \gamma_2 \geq 0$ *with* $\gamma_1^s + \gamma_2^s = 1$ *(*$\gamma_1 + \gamma_2 = 1$ *).*

The following theorem was proved by Dragomir and Fitzpatrick (1999):

Theorem 1.1. *Consider $h\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function where $h \in GK_s^2$, $0 < s < 1$ and $w_1, w_2 \in \mathbb{R}_+$, $w_1 < w_2$. If $h \in L^1([w_1, w_2])$, then*

$$2^{s-1} h\left(\frac{w_1 + w_2}{2}\right) \leq \frac{1}{w_2 - w_1}\int_{w_1}^{w_2} h(x)dx \leq \frac{h(w_1) + h(w_2)}{s+1}. \tag{3}$$

Theorem 1.2. *Consider $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+^{\alpha}$ is a function where $f \in GK_s^2$, $0 < s < 1$ and $w_1, w_2 \in \mathbb{R}_+$ with $w_1 < w_2$. If $f \in L^1([w_1, w_2])$, then*

$$\begin{aligned} 2^{\alpha(s-1)} f\left(\frac{w_1 + w_2}{2}\right) & \\ \leq \frac{\Gamma(1+\alpha)}{(w_2 - w_1)^{\alpha}} & \ {}_{w_1}I_{w_2}^{(\alpha)} f(x) \\ & \leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left(f(w_1) + f(w_2)\right). \end{aligned} \tag{4}$$

The proof of above theorem was started by Kılıçman and Saleh (2015a).

A mapping $h\colon [\kappa_1, \kappa_2] \times [z_1, z_2] \longrightarrow \mathbb{R}$ is a convex the co-ordinates on $[\kappa_1, \kappa_2] \times [z_1, z_2]$ where $\kappa_1 < \kappa_2$ and $z_1 < z_2$ if the next inequality:

$$h\left(\gamma\nu_1 + (1-\gamma)\nu_2, \gamma w_1 + (1-\gamma) w_2\right) \leq \gamma h(\nu_1, w_1) + (1-\gamma) f(\nu_2, w_2)$$

holds $\forall (\nu_1, w_1), (\nu_2, w_2) \in [\kappa_1, \kappa_2] \times [z_1, z_2]$ and $\gamma \in [0, 1]$.

Dragomir (2001) used the above defintion to obtain Hadamard's inequality for convex mapping on the co-ordinates on $[\mu_1, \mu_2] \times [\eta_1, \eta_2]$ from the plane:

Theorem 1.3. *Consider $h\colon [\mu_1, \mu_2] \times [\eta_1, \eta_2] \subseteq \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is a convex function on the co-ordinates on $[\mu_1, \mu_2] \times [\eta_1, \eta_2]$. Then the following inequalities hold*

$$\begin{aligned} & h\left(\frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}\right) \\ & \leq \frac{1}{(\mu_2 - \mu_1)(\eta_2 - \eta_1)} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} h(x_1, x_2) dx_1 dx_2 \\ & \leq \frac{h(\mu_1, \eta_1) + h(\mu_2, \eta_1) + h(\mu_1, \eta_2) + h(\mu_2, \eta_2)}{4}. \end{aligned} \tag{5}$$

Co-ordinated s-convex function of the second sense was defined such as (Alomari and Darus (2008a,b)):

Definition 1.2. *A mapping $h\colon [\omega_1, \omega_2] \times [z_1, z_2] \longrightarrow \mathbb{R}$ is s-convex in the first sense (in the second sense) on $[\omega_1, \omega_2] \times [z_1, z_2]$ where $\omega_1 < \omega_2$ and $z_1 < z_2$ if this inequality : $h(\gamma_1 x_1 + \gamma_2 x_2, \gamma_1 y_1 + \gamma_2 y_2) \leq \gamma_1^s h(x_1, y_1) + \gamma_2^s f(x_2, y_2)$ holds, $\forall (x_1, y_1), (x_2, y_2) \in [\omega_1, \omega_2] \times [z_1, z_2]$;$\forall \gamma_1, \gamma_2 \geq 0$ with $\gamma_1^s + \gamma_2^s = 1 (\gamma_1 + \gamma_2 = 1)$ and for some fixed $s \in (0, 1]$.*

The next theorem was proved by Alomari and Darus (2008b) :

Theorem 1.4. *Consider* $h\colon [\kappa_1,\kappa_2]\times[z_1,z_2]\subseteq \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ *is a* s*-convex function (in the second sense) on the co-ordinates on* $[\kappa_1,\kappa_2]\times[z_1,z_2]$*. Then*

$$\begin{aligned} 4^{s-1}h\left(\frac{\kappa_1+\kappa_2}{2},\frac{z_1+z_2}{2}\right) &\\ \leq \frac{1}{(\kappa_2-\kappa_1)(z_2-z_1)}\int_{\kappa_1}^{\kappa_2}\int_{z_1}^{z_2} h(x_1,x_2)dx_1dx_2 &\\ \leq \frac{h(\kappa_1,z_1)+h(\kappa_2,z_1)+h(\kappa_1,z\kappa_2)+h(\kappa_2,z_2)}{(s+1)^2}. & \end{aligned} \tag{6}$$

One of most important definitions in this paper is (see Kılıçman and Saleh (2015b)):

Definition 1.3. *Suppose that* $f\colon [a_1,a_2]\times[b_1,b_2]\subseteq\mathbb{R}^2_+\longrightarrow\mathbb{R}^\alpha$ *is a mapping with* $a_1<a_2$ *and* $b_1<b_2$*, then*

(i) f *is called generalized* s*-convex* $(0<s<1)$ *in the first sense which is denoted by* GE^1_s *(in the second sense which is denoted by* GE^2_s *) on* $[a_1,a_2]\times[b_1,b_2]$ *if* $f(\gamma_1x_1+\gamma_2x_2,\gamma_1y_1+\gamma_2y_2)\leq\gamma_1^{s\alpha}f(x_1,y_1)+\gamma_2^{s\alpha}f(x_2,y_2)$ *holds* $\forall(x_1,y_1),(x_2,y_2)\in[a_1,a_2]\times[b_1,b_2]$*;* $\forall\gamma_1,\gamma_2\geq 0$ *with* $\gamma_1^s+\gamma_2^s=1$ $(\gamma_1+\gamma_2=1)$*.*

(ii) f *is called generalized* s*-convex* $(0<s<1)$ *in the first sense wich is denoted by* $GK^1_{s_1,s_2}$ *(in the second sense wich is denoted by* $GK^2_{s_1,s_2}$*) on* $[a_1,a_2]\times[b_1,b_2]$*, if there exists* $s_1,s_2\in(0,1)$ *with* $s=\frac{s_1+s_2}{2}$*, such that*

$$f(\gamma_1x_1+\gamma_2x_2,\gamma_1y_1+\gamma_2y_2)\leq\gamma_1^{s_1\alpha}f(x_1,y_1)+\gamma_2^{s_2\alpha}f(x_2,y_2) \tag{7}$$

holds $\forall(x_1,y_1),(x_2,y_2)\in[a_1,a_2]\times[b_1,b_2]$*;* $\forall\gamma_1,\gamma_2\geq 0$ *with* $\gamma_1^{s_1}+\gamma_2^{s_2}=1$ $(\gamma_1+\gamma_2=1)$ *and for all fixed* $0<s_1,s_2<1$*.*

The following results can be found in (Kirmaci et al. (2007)):

Theorem 1.5. *Suppose that* $h_1,h_2\colon[\mu_1,\mu_2]\subseteq\mathbb{R}_+\longrightarrow\mathbb{R},\mu_1,\mu_2\in\mathbb{R}_+,\mu_1<\mu_2$ *are functions where* $h_2,h_1h_2\in L^1([\mu_1,\mu_2])$*. If* h_1 *is non-negative convex function on* $[\mu_1,\mu_2]$ *and* h_2 *is* s*-convex on* $[\mu_1,\mu_2]$*, for some* $0<s<1$*, then*

$$\begin{aligned} 2^sh_1\left(\frac{\mu_1+\mu_2}{2}\right)h_2\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}h_1(x)h_2(x)dx &\\ \leq\frac{1}{(s+1)(s+2)}T_1(\mu_1,\mu_2)+\frac{1}{s+2}T_2(\mu_1,\mu_2) & \end{aligned} \tag{8}$$

where

$$T_1(\mu_1,\mu_2)=h_1(\mu_1)h_2(\mu_1)+h_1(\mu_2)h_2(\mu_2)$$

and

$$T_2(\mu_1,\mu_2)=h_1(\mu_1)h_2(\mu_2)+h_1(\mu_2)h_2(\mu_1).$$

Theorem 1.6. *Suppose that $h_1,h_2\colon[\mu_1,\mu_2]\subseteq\mathbb{R}_+\longrightarrow\mathbb{R},\mu_1,\mu_2\in\mathbb{R}_+,\mu_1<\mu_2$ are functions where $h_2,h_1h_2\in L^1([\mu_1,\mu_2])$. If h_1 is non-negative convex function on $[\mu_1,\mu_2]$ and h_2 is s-convex on $[\mu_1,\mu_2]$, for some $0<s<1$, then*

$$\begin{aligned}&\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}h_1(x)h_2(x)dx\\&\leq\frac{1}{s+2}T_1(\mu_1,\mu_2)+\frac{1}{(s+1)(s+2)}T_2(\mu_1,\mu_2)\end{aligned}\tag{9}$$

where $T_1(\mu_1,\mu_2)$ and $T_2(\mu_1,\mu_2)$ are defined in Theorem 1.5 .

Theorem 1.7. *Suppose that $h_1,h_2\colon[\mu_1,\mu_2]\subseteq\mathbb{R}_+\longrightarrow\mathbb{R},\mu_1,\mu_2\in\mathbb{R}_+,\mu_1<\mu_2$ are functions where $h_1,h_2,h_1h_2\in L^1([\mu_1,\mu_2])$. If h_1 is s_1-convex and h_2 is s_2-convex on $[\mu_1,\mu_2]$, for some fixed $0<s_1,s_2<1$, then*

$$\begin{aligned}&\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}h_1(x)h_2(x)dx\\&\leq\frac{1}{s_1+s_2+1}T_1(\mu_1,\mu_2)+\beta(s_1+1,s_2+1)T_2(\mu_1,\mu_2),\end{aligned}\tag{10}$$

where $T_1(\mu_1,\mu_2)$ and $T_2(\mu_1,\mu_2)$ are defined in Theorem 1.5.

2. Inequalities for Products of Two Generalized Functions

In this section, some new inequalities for products of two functions on fractal sets are established.

Theorem 2.1. *Suppose that $h_1,h_2\colon[\omega_1,\omega_2]\longrightarrow\mathbb{R}^\alpha,\omega_1,\omega_2\in\mathbb{R}_+,\omega_1<\omega_2$ are functions where $h_2,h_1h_2\in L^1([\omega_1,\omega_2])$. If h_1 is a non-negative generalized convex function on $[\omega_1,\omega_2]$ and $h_2\in GK_s^2$ on $[\omega_1,\omega_2]$ for some fixed $0<s\leq1$, then*

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1(x)h_2(x)(dx)^\alpha\\&\leq\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}T_1(\omega_1,\omega_2)+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}\right]T_2(\omega_1,\omega_2),\end{aligned}\tag{11}$$

where

$$T_1(\omega_1, \omega_2) = h_1(\omega_1)h_2(\omega_1) + h_1(\omega_2)h_2(\omega_2)$$

and

$$T_2(\omega_1, \omega_2) = h_1(\omega_1)h_2(\omega_2) + h_1(\omega_2)h_2(\omega_1).$$

Proof. Since h_1 is a generalized convex and $h_2 \in GK_s^2$ on $[w_1, w_2]$ and they are non-negative, then

$$\begin{aligned} &h_1(\gamma w_1 + (1-\gamma)w_2)h_2(\gamma w_1 + (1-\gamma)w_2) \\ &\qquad \le \gamma^{\alpha(s+1)}h_1(\omega_1)h_2(\omega_2) + \gamma^{\alpha}(1-\gamma)^{\alpha s}h_1(\omega_1)h_2(\omega_2) \\ &\qquad +\gamma^{\alpha s}(1-\gamma)^{\alpha}h_1(w_2)h_2(w_1) + (1-\gamma)^{\alpha(s+1)}h_1(w_2)h_2(w_2), \forall \gamma \in [0,1] \end{aligned}$$

therefore,

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_0^1 h_1(\gamma w_1 + (1-\gamma)w_2)h_2(\gamma w_1 + (1-\gamma)w_2)(d\gamma)^{\alpha} \\ &\le \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}h_1(w_1)h_2(w_1) + \left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} - \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}\right]h_1(w_1)h_2(w_2) \\ &+ \left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} - \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}\right]h_1(w_2)h_2(w_1) + \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}h_1(w_2)h_2(w_2). \end{aligned}$$

This completes the proof. □

Remark 2.1. *1. As aspecial case when $\alpha = 1$ in Theorem 2.1, then we get inequality (9) .*

2. *If one choose $\alpha = 1$ and $h_1 : [w_1, w_2] \longrightarrow \mathbb{R}$ where $h_1(x) = 1$, $\forall x \in [w_1, w_2]$ in Theorem 2.1, then we have the right hand side of (3).*

3. *If $h_1 : [w_1, w_2] \longrightarrow \mathbb{R}^{\alpha}$ where $h_1(x) = 1^{\alpha}$, $\forall x \in [w_1, w_2]$ in Theorem 2.1, then we have the right hand side of (4).*

The following function will be used in the next theorem $\beta_{\alpha}(x_1, x_2) = \int_0^1 \gamma^{\alpha(x_1-1)}(1-\gamma)^{\alpha(x_2-1)}(d\gamma)^{\alpha}, x_1, x_2 > 0.$

Theorem 2.2. *Suppose that $h_1, h_2 : [\omega_1, \omega_2] \longrightarrow \mathbb{R}^{\alpha}, \omega_1, \omega_2 \in \mathbb{R}_+, \omega_1 < \omega_2$ are functions where $h_1, h_2, h_1h_2 \in L^1([\omega_1, \omega_2])$. If $h_1 \in GK_{s_1}^2$ and $h_2 \in GK_{s_2}^2$ on $[w_1, w_2]$ for some fixed $0 < s_1, s_2 \le 1$, then*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\frac{1}{(w_2-w_1)^{\alpha}}\int_{w_1}^{w_2} h_1(x)h_2(x)(dx)^{\alpha} \\ &\le \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}T_1(w_1, w_2) + \frac{1}{\Gamma(1+\alpha)}\beta_{\alpha}(s_1+1, s_2+1)T_2(w_1, w_2), \end{aligned}$$

where $T_1(w_1, w_2)$ and $T_2(w_1, w_2)$ are defined in Theorem 2.1.

Proof. Since $h_1 \in GK^2_{s_1}$ and $h_2 \in GK^2_{s_2}$ on $[w_1, w_2]$ and they are non-negative, then

$$\begin{aligned} &h_1(\gamma\omega_1 + (1-\gamma)\omega_2)h_2(\gamma\omega_1 + (1-\gamma)\omega_2) \\ &\quad \le \gamma^{\alpha(s_1+s_2)}h_1(\omega_1)h_2(\omega_1) + \gamma^{\alpha s_1}(1-\gamma)^{\alpha s_2}h_1(\omega_1)h_2(\omega_2) \\ &\qquad + \gamma^{\alpha s_2}(1-\gamma)^{\alpha s_1}h_1(\omega_2)h_2(\omega_1) + (1-\gamma)^{\alpha(s_1+s_2)}h_1(\omega_2)h_2(\omega_2), \forall \gamma \in [0,1]. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\Gamma(1+\alpha)} \int_0^1 & h_1(\gamma\omega_1 + (1-\gamma)\omega_2)h_2(\gamma\omega_1 + (1-\gamma)\omega_2)(d\gamma)^\alpha \\ &\le \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\left[h_1(\omega_1)h_2(\omega_1) + h_1(\omega_2)h_2(\omega_2)\right] \\ &+ \frac{1}{\Gamma(1+\alpha)}h_1(\omega_1)h_2(\omega_2)\beta_\alpha(s_1+1, s_2+1) \\ &+ \frac{1}{\Gamma(1+\alpha)}h_1(\omega_2)h_2(\omega_1)\beta_\alpha(s_2+1, s_1+1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \frac{1}{(\omega_2-\omega_1)^\alpha} \int_{\omega_1}^{\omega_2} f_1(x)f_2(x)(dx)^\alpha \\ &\quad \le \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}T_1(\omega_1,\omega_2) + \frac{1}{\Gamma(1+\alpha)}\beta_\alpha(s_1+1, s_2+1)T_2(\omega_1,\omega_2). \end{aligned} \tag{12}$$

The proof is complete.

□

Remark 2.2. *As aspecial case when $\alpha = 1$ in the above theorem , then we obtain inequality (10).*

Theorem 2.3. *Suppose that $h_1, h_2 \colon [w_1, w_2] \longrightarrow \mathbb{R}^\alpha, w_1, w_2 \in \mathbb{R}_+, w_1 < w_2$ are functions such that $h_2, h_1h_2 \in L^1([w_1, w_2])$. If h_1 is a non-negative generalized convex function on $[w_1, w_2]$ and $h_2 \in GK^2_s$ on $[w_1, w_2]$ for some fixed $0 < s \le 1$,* *then*

$$\frac{1}{\Gamma(1+\alpha)}\left[h_1\left(\frac{\omega_1+\omega_2}{2}\right)h_2\left(\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$\left.-\frac{1}{2^{\alpha s}}\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1(x)h_2(x)(dx)^\alpha\right]$$
$$\leq\frac{1}{2^{\alpha s}}\left\{\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}\right]T_1(\omega_1,\omega_2)\right.$$
$$\left.+\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}T_2(\omega_1,\omega_2)\right\}, \tag{13}$$

where $T_1(\omega_1,\omega_2)$ and $T_2(\omega_1,\omega_2)$ are defined in Theorem 2.1.

Proof. Setting

$$\frac{w_1+w_2}{2}=\frac{\gamma w_1+(1-\gamma)w_2}{2}+\frac{(1-\gamma)w_1+\gamma w_2}{2},$$

which gives

$$h_1\left(\frac{w_1+w_2}{2}\right)h_2\left(\frac{w_1+w_2}{2}\right)$$
$$\leq\left(\frac{1}{2}\right)^{\alpha(s+1)}[h_1(\gamma w_1+(1-\gamma)w_2)h_2(\gamma w_1+(1-\gamma)w_2)$$
$$+f_1((1-\gamma)a_1+\gamma a_2)f_2((1-\gamma)a_1+\gamma w_2)]$$
$$+\left(\frac{1}{2}\right)^{\alpha(s+1)}\{[\gamma^\alpha h_1(w_1)+(1-\gamma)^\alpha h_1(w_2)]\left[(1-\gamma)^{\alpha s}h_2(w_1)+\gamma^{\alpha s}h_2(w_2)\right]$$
$$+\left[(1-\gamma)^\alpha h_1(w_1)+\gamma^\alpha h_1(w_2)\right]\left[\gamma^{\alpha s}h_2(w_1)+(1-\gamma)^{\alpha s}h_2(w_2)\right]\}.$$

Then

$$\frac{1}{\Gamma(1+\alpha)}h_1\left(\frac{w_1+w_2}{2}\right)h_2\left(\frac{w_1+w_2}{2}\right)$$
$$\leq\left(\frac{1}{2}\right)^{\alpha(s+1)}\frac{2^\alpha}{\Gamma(1+\alpha)}\frac{1}{(w_2-w_1)^\alpha}\int_{w_1}^{w_2}h_1(x)h_2(x)(dx)^\alpha$$
$$+\left(\frac{1}{2}\right)^{\alpha(s+1)}\left\{2^\alpha\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}\right]T_1(w_1,w_2)\right.$$
$$\left.+2^\alpha\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}T_2(w_1,w_2)\right\}.$$

$\square$

Remark 2.3. *1. If one choose $\alpha = 1$ in Theorem 2.3 , then we have inequality (8).*

2. When $\alpha = 1$ and $f_1 \colon [a_1, a_2] \longrightarrow \mathbb{R}$, where $f_1(x) = 1$, $\forall x \in [a_1, a_2]$ in Theorem 2.3, then

$$2^s f_2\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f_2(x)dx \leq \frac{f_2(a_1) + f_2(a_2)}{s+1}.$$

3. Some Generalized Hadamard-Type Inequalities

The next inequalities is considered as generalized Hadamard type inequalities connected with inequality (7) for $GK^2_{s_1,s_2}$

Theorem 3.1. *Let $f \colon [w_1, w_2] \times [z_1, z_2] \subseteq \mathbb{R}^2_+ \longrightarrow \mathbb{R}^\alpha_+$ be a function where $f \in GK^2_{s_1,s_2}$ on $[w_1, w_2] \times [z_1, z_2]$. Then one has the inequalities:*

$$\begin{aligned}
&\frac{4^{\alpha(s_2-1)} + 4^{\alpha(s_1-1)}}{2^\alpha} f\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right) \\
&\leq \frac{2^{\alpha(s_1-2)}}{(w_2-w_1)^\alpha}\Gamma(1+\alpha)\ {}_{w_1}I^{(\alpha)}_{w_2} f\left(x_1, \frac{z_1+z_2}{2}\right) \\
&\qquad + \frac{2^{\alpha(s_2-2)}}{(z_2-z_1)^\alpha}\Gamma(1+\alpha)\ {}_{z_1}I^{(\alpha)}_{z_2} f\left(\frac{z_1+z_2}{2}, x_2\right) \\
&\leq \frac{1}{(w_2-w_1)^\alpha(z_2-z_1)^\alpha}\int_{w_1}^{w_2}\int_{z_1}^{z_2} f(x_1,x_2)(dx_1)^\alpha(dx_2)^\alpha \\
&\leq \frac{\Gamma(1+s_1\alpha)\Gamma(1+\alpha)}{2^\alpha\Gamma(1+(s_1+1)\alpha)}\frac{\Gamma(1+\alpha)}{(w_2-w_1)^\alpha}\left[\ {}_{w_1}I^{(\alpha)}_{w_2} f(x_1,z_1) + {}_{w_1}I^{(\alpha)}_{w_2} f(x_1,z_2)\right] \\
&\qquad + \frac{\Gamma(1+s_2\alpha)\Gamma(1+\alpha)}{2^\alpha\Gamma(1+(s_2+1)\alpha)}\frac{\Gamma(1+\alpha)}{(z_2-z_1)^\alpha}\left[\ {}_{z_1}I^{(\alpha)}_{z_2} f(w_1,x_2) + {}_{z_1}I^{(\alpha)}_{z_2} f(w_2,x_2)\right] \\
&\leq \left(\frac{1}{2}\right)^\alpha\left\{\left[\frac{\Gamma(1+s_1\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_1+1)\alpha)}\right]^2 + \left[\frac{\Gamma(1+s_2\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_2+1)\alpha)}\right]^2\right\} \\
&\qquad (f(w_1,z_1) + f(w_1,z_2) + f(w_2,z_1) + f(w_2,z_2)). \qquad (14)
\end{aligned}$$

The above inequalities are sharp.

Proof. Because of $f \in GK^2_{s_1,s_2}$ on $[w_1, w_2] \times [z_1, z_2]$ if follows that the mapping $f_{x_1} : [z_1, z_2] \longrightarrow \mathbb{R}^{\alpha}_{+}$, $f_{x_1}(x_2) = f(x_1, x_2)$ is a generalized s_1-convex on $[z_1, z_2]$, $\forall x_1 \in [w_1, w_2]$ with $0 < s_1 < 1$, then by appling (4), we get

$$\begin{aligned} 2^{\alpha(s_1-1)} f_{x_1}\left(\frac{z_1+z_2}{2}\right) & \\ &\leq \frac{\Gamma(1+\alpha)}{(z_2-z_1)^{\alpha}} \quad {}_{z_1}I^{(\alpha)}_{z_2} f_{x_1}(x_2) \\ &\qquad \leq \frac{\Gamma(1+s_1\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_1+1)\alpha)}(f_{x_1}(z_1)+f_{x_1}(z_2)). \end{aligned} \tag{15}$$

Then

$$\begin{aligned} 2^{\alpha(s_1-1)} f\left(x_1, \frac{z_1+z_2}{2}\right) & \\ &\leq \frac{\Gamma(1+\alpha)}{(z_2-z_1)^{\alpha}} \quad {}_{z_1}I^{(\alpha)}_{z_2} f(x_1, x_2) \\ &\qquad \leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}(f(x_1,z_1)+f(x_1,z_2)), \forall x_1 \in [w_1, w_2]. \end{aligned}$$

That is

$$\begin{aligned} &\frac{2^{\alpha(s_1-1)}}{(w_2-w_1)^{\alpha}}\Gamma(1+\alpha) \quad {}_{w_1}I^{(\alpha)}_{w_2} f\left(x_1, \frac{z_1+z_2}{2}\right) \\ &\leq \frac{1}{(w_2-w_1)^{\alpha}(z_2-z_1)^{\alpha}} \int_{w_1}^{w_2}\int_{z_1}^{z_2} f(x_1,x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\leq \frac{\Gamma(1+s_1\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_1+1)\alpha)}\left[\frac{\Gamma(1+\alpha)}{(w_2-w_1)^{\alpha}} \quad {}_{w_1}I^{(\alpha)}_{w_2} f(x_1, z_1)\right. \\ &\qquad \left. + \frac{\Gamma(1+\alpha)}{(w_2-w_1)^{\alpha}} \quad {}_{w_1}I^{(\alpha)}_{w_2} f(x_1, z_2)\right]. \end{aligned} \tag{16}$$

A similar arguments applied for the mapping $f_{x_2} : [w_1, w_2] \longrightarrow \mathbb{R}^{\alpha}_{+}$, $f_{w_2}(x_1) = f(x_1, x_2)$ is a generalized s_2-convex on $[w_1, w_2]$ for all $x_2 \in [z_1, z_2]$ with $s_2 \in (0, 1)$.

Then, it follows that :

$$\begin{aligned}
&\frac{2^{\alpha(s_2-1)}}{(z_2-z_1)^{\alpha}}\Gamma(1+\alpha) \quad {}_{z_1}I_{z_2}^{(\alpha)} f\left(\frac{w_1+w_2}{2}, x_2\right) \\
&\leq \frac{1}{(z_2-z_1)^{\alpha}(w_2-w_1)^{\alpha}} \int_{z_1}^{z_2}\int_{w_1}^{w_2} f(x_1,x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\
&\leq \frac{\Gamma(1+s_2\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_2+1)\alpha)}\left[\frac{\Gamma(1+\alpha)}{(z_2-z_1)^{\alpha}} \quad {}_{z_1}I_{z_2}^{(\alpha)} f(w_1,x_2) \right. \\
&\qquad \left. +\frac{\Gamma(1+\alpha)}{(z_2-z_1)^{\alpha}} \quad {}_{z_1}I_{z_2}^{(\alpha)} f(w_2,x_2)\right].
\end{aligned} \tag{17}$$

Summing inequalities (16) and (17), give the second and the third inequalities in (14).

Then, by (4) , we obtain

$$\frac{4^{\alpha(s_2-1)}}{2^{\alpha}} f\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right) \leq \frac{2^{\alpha(s_2-2)}}{(z_2-z_1)^{\alpha}}\Gamma(1+\alpha) \quad {}_{z_1}I_{z_2}^{(\alpha)} f\left(\frac{w_1+w_2}{2}, x_2\right) \tag{18}$$

and

$$\frac{4^{\alpha(s_1-1)}}{2^{\alpha}} f\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right) \leq \frac{2^{\alpha(s_1-2)}}{(w_2-a_1)^{\alpha}}\Gamma(1+\alpha) \quad {}_{w_1}I_{w_2}^{(\alpha)} f\left(x_1, \frac{z_1+z_2}{2}\right). \tag{19}$$

Adding (18) and (19) we get the first inequality in (14).

Here, we find

$$\frac{\Gamma(1+\alpha)}{(w_2-w_1)^{\alpha}} \quad {}_{w_1}I_{w_2}^{(\alpha)} f(x_1,z_1) \leq \frac{\Gamma(1+s_1\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_1+1)\alpha)}\left(f(w_1,z_1)+f(w_2,z_1)\right)$$

$$\frac{\Gamma(1+\alpha)}{(w_2-w_1)^{\alpha}} \quad {}_{w_1}I_{w_2}^{(\alpha)} f(x_1,z_2) \leq \frac{\Gamma(1+s_1\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_1+1)\alpha)}\left(f(w_1,z_2)+f(w_2,z_2)\right)$$

$$\frac{\Gamma(1+\alpha)}{(z_2-z_1)^{\alpha}} \quad {}_{z_1}I_{z_2}^{(\alpha)} f(w_1,x_2) \leq \frac{\Gamma(1+s_2\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_2+1)\alpha)}\left(f(w_1,z_1)+f(w_1,z_2)\right)$$

$$\frac{\Gamma(1+\alpha)}{(z_2-z_1)^{\alpha}} \quad {}_{z_1}I_{z_2}^{(\alpha)} f(w_2,x_2) \leq \frac{\Gamma(1+s_2\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s_2+1)\alpha)} \left(f(w_2,z_1)+f(w_2,z_2)\right).$$

Adding the above inequalities we get the last inequality in (14). □

Remark 3.1. *1. As aspecial case when $s_1 = s_2 = 1$, and $\alpha = 1$ in (14), then this inequalit reduced to inequality (5).*

2. Similarly, in (14), if $s_1 = s_2$ and $\alpha = 1$, then the inequality (14) is redaced to inequality (6).

4. Inequalities for Product of Two Generalized Functions

In this section, we give some new inequalities for product of two functions on the co-ordinates on fractal sets.

Theorem 4.1. *Let $h_1, h_2 \colon [\omega_1, \omega_2] \times [z_1, z_2] \subseteq \mathbb{R}_+^2 \longrightarrow \mathbb{R}^{\alpha}, \omega_1 < \omega_2, z_1 < z_2$ be functions where $h_1, h_2, h_1h_2 \in L^2([\omega_1, \omega_2] \times [z_1, z_2])$. If h_1 is a non-negative generalized convex on the co-ordinates on $[\omega_1, \omega_2] \times [z_1, z_2]$ and $h_2 \in GK^2_{s_1,s_2}$ on $[\omega_1, \omega_2] \times [z_1, z_2]$, $\forall 0 < s_1, s_2 < 1$, such that $s = \frac{s_1+s_2}{2}$, then one has the inequality:*

$$\begin{aligned} \frac{1}{(\Gamma(1+\alpha))^2} \frac{1}{(\omega_2-\omega_1)^{\alpha}(z_2-z_1)^{\alpha}} \int_{\omega_1}^{\omega_2} \int_{z_1}^{z_2} h_1(x_1,x_2)h_2(x_1,x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ \leq \left(\frac{1}{2}\right)^{\alpha} \big\{(p_1^2+p_2^2)T_1(\omega_1,\omega_2,z_1,z_2) \\ +(p_1q_1+p_2q_2)T_2(\omega_1,\omega_2,z_1,z_2) \\ +(q_1^2+q_2^2)T_3(\omega_1,\omega_2,z_1,z_2)\big\} \end{aligned} \tag{20}$$

where

$$\begin{aligned} T_1(\omega_1,\omega_2,z_1,z_2) &= h_1(\omega_1,z_1)h_2(\omega_1,z_1)+h_1(\omega_2,z_1)h_2(\omega_2,z_1) \\ &\quad +h_1(\omega_1,z_2)h_2(\omega_1,z_2)+h_1(\omega_2,z_2)h_2(\omega_2,z_2), \end{aligned}$$

$$\begin{aligned} T_2(\omega_1,\omega_2,z_1,z_2) &= h_1(\omega_1,z_1)h_2(\omega_2,z_1)+h_1(\omega_2,z_1)h_2(\omega_1,z_1) \\ &\quad +h_1(\omega_1,z_2)h_2(\omega_2,z_2)+h_1(\omega_2,z_2)h_2(\omega_1,z_2) \\ &\quad +h_1(\omega_1,z_1)h_2(\omega_1,z_2)+h_1(\omega_2,z_1)h_2(\omega_2,z_2) \\ &\quad +h_1(\omega_1,z_2)h_2(\omega_1,z_1)+h_1(\omega_2,z_2)h_2(\omega_2,z_1), \end{aligned}$$

$$T_3(\omega_1,\omega_2,z_1,z_2) = h_1(\omega_1,z_1)h_2(\omega_2,z_2)+h_1(\omega_2,z_1)h_2(\omega_1,z_2) + h_1(\omega_1,z_2)h_2(\omega_2,z_1)+h_1(\omega_2,z_2)h_2(\omega_1,z_1)$$

and

$$p_1 = \frac{\Gamma(1+(s_2+1)\alpha)}{\Gamma(1+(s_2+2)\alpha)}, \quad q_1 = \frac{\Gamma(1+s_2\alpha)}{\Gamma(1+(s_2+1)\alpha)} - \frac{\Gamma(1+(s_2+1)\alpha)}{\Gamma(1+(s_2+2)\alpha)}$$

$$p_2 = \frac{\Gamma(1+(s_1+1)\alpha)}{\Gamma(1+(s_1+2)\alpha)}, \quad q_2 = \frac{\Gamma(1+s_1\alpha)}{\Gamma(1+(s_1+1)\alpha)} - \frac{\Gamma(1+(s_1+1)\alpha)}{\Gamma(1+(s_1+2)\alpha)}.$$

Proof. Because h_1 is a generalized convex function and $h_2 \in GK^2_{s_1,s_2}$ on $[\omega_1,\omega_2]\times[z_1,z_2]$. Therefore, the partial mappings h_{1x_2} and h_{1x_1} are generalized convex and non-negative on $[w_1,w_2]$ and $[z_1,z_2]$, respectively. Also, h_{2x_2} and h_{2x_1} are generalized s_1-,s_2- convex on $[\omega_1,\omega_2]$ and $[z_1,z_2]$, respectively, $\forall x_1 \in [\omega_1,\omega_2], x_2 \in [z_1,z_2]$, $\forall s_1,s_2 \in (0,1)$, such that $s=\frac{s_1+s_2}{2}$. Now, by applying $h_{1x_1}(x_2)h_{2x_1}(x_2)$ to (11) on $[z_1,z_2]$, we get

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(w_2-w_1)^\alpha(z_2-z_1)^\alpha}\int_{w_1}^{w_2}\int_{z_1}^{z_2} h_1(x_1,x_2)h_2(x_1,x_2)(dx_1)^\alpha(dx_2)^\alpha \\
&\le p_1\left[\frac{1}{\Gamma(1+\alpha)}\frac{1}{(w_2-w_1)^\alpha}\int_{w_1}^{w_2} h_1(x_1,z_1)h_2(x_1,z_1)(dx_1)^\alpha \right.\\
&\qquad \left. +\frac{1}{\Gamma(1+\alpha)}\frac{1}{(w_2-w_1)^\alpha}\int_{w_1}^{w_2} h_1(x_1,z_2)h_2(x_1,z_2)(dx_1)^\alpha\right] \\
&+q_1\left[\frac{1}{\Gamma(1+\alpha)}\frac{1}{(w_2-w_1)^\alpha}\int_{w_1}^{w_2} h_1(x_1,z_1)h_2(x_1,z_2)(dx_1)^\alpha \right.\\
&\qquad \left. +\frac{1}{\Gamma(1+\alpha)}\frac{1}{(w_2-w_1)^\alpha}\int_{w_1}^{w_2} h_1(x_1,z_2)h_2(x_1,z_1)(dx_1)^\alpha\right].
\end{aligned} \tag{21}$$

Now, by appling (11) in (21) and a combination the result and (21), we get

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(w_2-w_1)^\alpha(z_2-z_1)^\alpha}\int_{w_1}^{w_2}\int_{z_1}^{z_2} h_1(x_1,x_2)h_2(x_1,x_2)(dx_1)^\alpha(dx_2)^\alpha \\
&\qquad \le p_1^2T_1(\omega_1,\omega_2,z_1,z_2)+p_1q_1T_2(\omega_1,\omega_2,z_1,z_2)+q_1^2T_3(\omega_1,\omega_2,z_1,z_2).
\end{aligned} \tag{22}$$

Similarly , if we apply $h_{1_{x_2}}(x_1)h_{2_{x_2}}(x_1)$ to (11) on $[\omega_1, \omega_2]$, we get

$$\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(w_2-w_1)^\alpha(z_2-z_1)^\alpha}\int_{w_1}^{w_2}\int_{z_1}^{z_2} h_1(x_1,x_2)h_2(x_1,x_2)(dx_1)^\alpha(dx_2)^\alpha$$
$$\leq \left(\frac{1}{2}\right)^\alpha \left\{(p_1^2+p_2^2)T_1(w_1,w_2,z_1,z_2) +(p_1q_1+p_2q_2)T_2(w_1,w_2,z_1,z_2) +(q_1^2+q_2^2)T_3(w_1,w_2,z_1,z_2)\right\}.$$

□

Remark 4.1. If we take $\alpha = 1$ and $s = s_1 = s_2$, then we get

$$\frac{1}{(w_2-w_1)(z_2-z_1)}\int_{w_1}^{w_2}\int_{z_1}^{z_2} f_1(x_1,x_2)f_2(x_1,x_2)(dx_1)(dx_2)$$
$$\leq \frac{1}{(s+2)^2}T_1(w_1,w_2,z_1,z_2) +\frac{1}{(s+1)(s+2)^2}T_2(w_1,w_2,z_1,z_2) +\frac{1}{(s+1)^2(s+2)^2}T_3(w_1,w_2,z_1,z_2),$$

which was proved by Akdemir and Özdemir (2015).

Theorem 4.2. *Suppose that $h_1, h_2 \colon [\omega_1, \omega_2] \times [z_1, z_2] \subseteq \mathbb{R}_+^2 \longrightarrow \mathbb{R}^\alpha, \omega_1 < \omega_2, z_1 < z_2$ are functions such that $h_1, h_2, h_1h_2 \in L^2([\omega_1, \omega_2] \times [z_1, z_2])$. If h_1 is a non-negative generalized convex on the co-ordinates on $[\omega_1, \omega_2] \times [z_1, z_2]$ and $h_2 \in GK^2_{s_1,s_2}$ on $[w_1, w_2] \times [z_1, z_2]$, $\forall 0 < s_1, s_2 < 1$, such that $s = \frac{s_1+s_2}{2}$, then:*

$$2^{\alpha(s_1+s_2+1)}\frac{1}{(\Gamma(1+\alpha))^2}h_1\left(\frac{\omega_1+\omega_2}{2},\frac{z_1+z_2}{2}\right)h_2\left(\frac{\omega_1+\omega_2}{2},\frac{z_1+z_2}{2}\right)$$
$$\leq 2^\alpha\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(w_2-w_1)^\alpha(z_2-z_1)^\alpha}\int_{w_1}^{w_2}\int_{z_1}^{z_2} h_1(x_1,x_2)h_2(x_1,x_2)(dx_1)^\alpha(dx_2)^\alpha$$
$$+(q_1^2+q_2^2+2^\alpha q_2p_1+2^\alpha q_1p_2)T_1(\omega_1,\omega_2,z_1,z_2) +(p_1q_1+p_2q_2+2^\alpha q_2q_1+2^\alpha p_1p_2)T_2(\omega_1,\omega_2,z_1,z_2) +(p_1^2+p_2^2+2^\alpha p_2q_1+2^\alpha p_1q_2)T_3(\omega_1,\omega_2,z_1,z_2)$$

where $T_1(\omega_1, \omega_2, z_1, z_2)$,$T_2(\omega_1, \omega_2, z_1, z_2)$, $T_3(\omega_1, \omega_2, z_1, z_2)$, p_1, q_1, p_2 and q_2 are defined in Theorem 4.1.

Proof. Applying

$$2^{\alpha s_1} h_1\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right) h_2\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right)$$

and

$$2^{\alpha s_2} h_1\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right) h_2\left(\frac{w_1+w_2}{2}, \frac{z_1+z_2}{2}\right)$$

to (13) and multiplying both sides by $2^{\alpha s_2}$ and and $2^{\alpha s_1}$, respectively, then dividing both sides by $\Gamma(1+\alpha)$. After that adding the results

$$\begin{aligned}
&2^{\alpha(s_1+s_2+1)} \frac{1}{(\Gamma(1+\alpha))^2} h_1\left(\frac{\omega_1+\omega_2}{2}, \frac{z_1+z_2}{2}\right) h_2\left(\frac{\omega_1+\omega_2}{2}, \frac{z_1+z_2}{2}\right) \\
&\leq 2^{\alpha s_2} \frac{1}{(\Gamma(1+\alpha))^2} \frac{1}{(a_2-a_1)^\alpha} \int_{\omega_1}^{\omega_2} h_1\left(x_1, \frac{z_1+z_2}{2}\right) h_2\left(x_1, \frac{z_1+z_2}{2}\right) (dx_1)^\alpha \\
&+2^{\alpha s_1} \frac{1}{(\Gamma(1+\alpha))^2} \frac{1}{(z_2-z_1)^\alpha} \int_{z_1}^{z_2} h_1\left(\frac{\omega_1+\omega_2}{2}, x_2\right) h_2\left(\frac{\omega_1+\omega_2}{2}, x_2\right) (dx_2)^\alpha \\
&+q_2\left[2^{\alpha s_2} \frac{1}{\Gamma(1+\alpha)} h_1\left(\omega_1, \frac{z_1+z_2}{2}\right) h_2\left(\omega_1, \frac{z_1+z_2}{2}\right)\right. \\
&\qquad \left. +2^{\alpha s_2} \frac{1}{\Gamma(1+\alpha)} h_1\left(\omega_2, \frac{z_1+z_2}{2}\right) h_2\left(\omega_2, \frac{z_1+z_2}{2}\right)\right] \\
&+p_2\left[2^{\alpha s_2} \frac{1}{\Gamma(1+\alpha)} h_1\left(\omega_1, \frac{z_1+z_2}{2}\right) h_2\left(\omega_2, \frac{z_1+z_2}{2}\right)\right. \\
&\qquad \left. +2^{\alpha s_2} \frac{1}{\Gamma(1+\alpha)} h_1\left(\omega_2, \frac{z_1+z_2}{2}\right) h_2\left(\omega_1, \frac{z_1+z_2}{2}\right)\right] \\
&+q_1\left[2^{\alpha s_1} \frac{1}{\Gamma(1+\alpha)} h_1\left(\frac{\omega_1+\omega_2}{2}, z_1\right) h_2\left(\frac{\omega_1+\omega_2}{2}, z_1\right)\right. \\
&\qquad \left. +2^{\alpha s_1} \frac{1}{\Gamma(1+\alpha)} h_1\left(\frac{\omega_1+\omega_2}{2}, z_2\right) h_2\left(\frac{\omega_1+\omega_2}{2}, z_2\right)\right] \\
&+p_1\left[2^{\alpha s_1} \frac{1}{\Gamma(1+\alpha)} h_1\left(\frac{\omega_1+\omega_2}{2}, z_1\right) h_2\left(\frac{\omega_1+\omega_2}{2}, z_2\right)\right. \\
&\qquad \left. +2^{\alpha s_1} \frac{1}{\Gamma(1+\alpha)} h_1\left(\frac{\omega_1+\omega_2}{2}, z_2\right) h_2\left(\frac{\omega_1+\omega_2}{2}, z_1\right)\right].
\end{aligned} \tag{23}$$

Thus, applying (13) to each term within the brackets and a combination of the result inequalities and (23), we get

$$\begin{aligned}
&2^{\alpha(s_1+s_2+1)}\frac{1}{(\Gamma(1+\alpha))^2}h_1\left(\frac{\omega_1+\omega_2}{2},\frac{z_1+z_2}{2}\right)h_2\left(\frac{\omega_1+\omega_2}{2},\frac{z_1+z_2}{2}\right)\\
&\le 2^{\alpha s_2}\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1\left(x_1,\frac{z_1+z_2}{2}\right)h_2\left(x_1,\frac{z_1+z_2}{2}\right)(dx_1)^\alpha\\
&+2^{\alpha s_1}\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(z_2-z_1)^\alpha}\int_{z_1}^{z_2}h_1\left(\frac{\omega_1+\omega_2}{2},x_2\right)h_2\left(\frac{\omega_1+\omega_2}{2},x_2\right)(dx_2)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}q_2\frac{1}{(z_2-z_1)^\alpha}\int_{z_1}^{z_2}h_1(\omega_1,x_2)h_2(\omega_1,x_2)(dx_2)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}q_2\frac{1}{(z_2-z_1)^\alpha}\int_{z_1}^{z_2}h_1(\omega_2,x_2)h_2(\omega_2,x_2)(dx_2)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}p_2\frac{1}{(z_2-z_1)^\alpha}\int_{z_1}^{z_2}h_1(\omega_1,x_2)h_2(\omega_2,x_2)(dx_2)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}p_2\frac{1}{(z_2-z_1)^\alpha}\int_{z_1}^{z_2}h_1(\omega_2,x_2)h_2(\omega_1,x_2)(dx_2)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}q_1\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1(x_1,z_1)h_2(x_1,z_1)(dx_1)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}q_1\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1(x_1,z_2)h_2(x_1,z_2)(dx_1)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}p_1\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1(x_1,z_1)h_2(x_1,z_2)(dx_1)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}p_1\frac{1}{(\omega_2-\omega_1)^\alpha}\int_{\omega_1}^{\omega_2}h_1(x_1,z_2)h_2(x_1,z_1)(dx_1)^\alpha\\
&+(q_1^2+q_2^2)T_1(\omega_1,\omega_2,z_1,z_2)\\
&\qquad+(p_1q_1+p_2q_2)T_2(\omega_1,\omega_2,z_1,z_2)\\
&\qquad\ +(p_1^2+p_2^2)T_3(\omega_1,\omega_2,z_1,z_2).
\end{aligned}\tag{24}$$

Now, by applying

$$2^{\alpha s_2}h_1\left(x_1,\frac{z_1+z_2}{2}\right)h_2\left(x_1,\frac{z_1+z_2}{2}\right)$$

and

$$2^{\alpha s_1}h_1\left(x_1,\frac{z_1+z_2}{2}\right)h_2\left(x_1,\frac{z_1+z_2}{2}\right)$$

to (13), integrating over $[\omega_1, \omega_2]$, $[z_1, z_2]$, respectively, then dividing both sides by $(\omega_2 - \omega_1)^\alpha$,$(z_2 - z_1)^\alpha$, respectively. After that by adding the results to (24), the proof is complete. □

5. Applications to Special Means

Here, consider the following generalized means: $A(a_1, a_2) = \frac{a_1^\alpha + a_2^\alpha}{2^\alpha}$, $K(a_1, a_2) = \left(\frac{a_1^{2\alpha} + a_2^{2\alpha}}{2^\alpha}\right)^{\frac{1}{2}}$ and $G(a_1, a_2) = (a_1^\alpha a_2^\alpha)^{\frac{1}{2}}$, $\quad a_1, a_2 \geq 0$. Next, this example was found in Mo and Sui (2014):

Let $0 < s < 1$ and $a_1^\alpha, a_2^\alpha, a_3^\alpha \in \mathbb{R}^\alpha$. Defining for $x \in \mathbb{R}_+$,

$$f(n) = \begin{cases} a_1^\alpha & n = 0; \\ a_2^\alpha n^{s\alpha} + a_3^\alpha & n > 0. \end{cases}$$

If $a_2^\alpha \geq 0^\alpha$ and $0^\alpha \leq a_3^\alpha \leq a_1^\alpha$, then $f \in GK_s^2$.

Proposition 5.1. *Let $a_1, a_2 \in \mathbb{R}_+, a_1 < a_2$ and $a_2 - a_1 \leq 1$, then :*

$$2^\alpha \left[\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] K^2(a_1, a_2) \leq \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - 2^\alpha \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] G^2(a_1, a_2).$$

Proof. If f_1 is a non-negative generalized convex function and f_2 is a generalized s-convex on $[a_1, a_2]$; then; in Theorem 2.1 if we choose $f_1, f_2 \colon [0, 1] \longrightarrow [0^\alpha, 1^\alpha]$, $f_1(x) = x^{2\alpha}$, $f_2(x) = x^{\alpha s}$, where $x \in [a_1, a_2]$ and $s = 1$, so

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \frac{1}{(a_2 - a_1)^\alpha} \int_{a_1}^{a_2} x^{3\alpha} (dx)^\alpha \\ &\quad \leq \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(a_2^{3\alpha} + a_1^{3\alpha}\right) + \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left(a_2^\alpha a_1^{2\alpha} + a_2^{2\alpha} a_1^\alpha\right) \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} &\frac{1}{(a_2 - a_1)^\alpha} \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \left(a_2^{4\alpha} - a_1^{4\alpha}\right) \\ &\quad \leq \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(a_2^{3\alpha} + a_1^{3\alpha}\right) + \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left(a_2^\alpha a_1^\alpha\right) \left(a_1^\alpha + a_2^\alpha\right). \end{aligned}$$

By a simple calculation, we obtain the required result. □

Proposition 5.2. *Let $a_1, a_2 \in \mathbb{R}_+, a_1 < a_2$ and $a_2 - a_1 \leq 1$, then :*

$$\frac{1}{2^\alpha \Gamma(1+\alpha)} A^2(a_1,a_2) + \left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right] K^2(a_1,a_2)$$
$$\leq \left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+\alpha)}{2^\alpha \Gamma(1+2\alpha)}\right] G^2(a_1,a_2)$$

Proof. The result follows from Theorem 2.3 with $f_1, f_2 \colon [0,1] \longrightarrow [0^\alpha, 1^\alpha]$, $f_1(x) = x^{2\alpha}$, $f_2(x) = x^{\alpha s}$ where $x \in [a_1, a_2]$ and $s = 1$. □

6. Conclusion

In this work, some new inequalities for product of generalized s-convex functions on the co-ordinates on $\mathbf{R}^\alpha$ have been proposed. Further, some of their applications have been presented.

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